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DISJOINTNESS AND CLASSES OF MINIMAL TRANSFORMATION GROUPS

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Disjointness and classes of minimal transformation groups

by

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ABSTRACT

We present disjointness relations between families of minimal topological transformation groups with a fixed arbitrary phase group T.

KEY WORDS & PHRASES: *Quasi-factor, disjointness*

0. INTRODUCTION

In the study of the structure of minimal topological transformation groups (ttg's for short) the concept of disjointness is a kind of external device. However, in dealing with this concept we need to know a lot about the internal structure of the ttg's under consideration. So in the Sections 1 and 3 of this paper emphasis is on the internal structure, and they don't bring many new facts. Section 1 is devoted to quasi-factors, disjointness, and repeats known facts which, except for Theorem 1.5, can be found in [1] and [10]. In Section 2 we prove that a minimal ttg is disjoint from every minimal weakly mixing ttg iff it has no nontrivial weakly mixing quasi-factors. Section 3 gives a treatment of towers of extensions which is a modification of [7],[8] and [9], but is similar to [4]. In Theorem 3.8 we show that a distal extension is the inverse limit of almost periodic extensions iff it is a PI extension. In the last section we establish disjointness of several classes and illustrate this with a table.

We assume basic knowledge about topological dynamics as can be found in [2] and [5], such as the notions of minimality, distality, proximality and almost periodicity, both for ttg's and homomorphisms of ttg's (continuous equivariant maps between the phase spaces). Often a homomorphism $\phi: X \to Y$ of minimal ttg's (as well as X itself) will be called an extension (of Y). The phase group T will be fixed and we put no restrictions on T. The phase spaces always are assumed to be compact T_2. The action of T on X will be written as a left product.

Let M be the universal minimal ttg for T. Then M is isomorphic to every minimal left ideal in the eveloping semigroup E(M) of M. Being a semigroup itself, M acts accordingly on every minimal ttg X, and $X = \{px \mid p \in M\}$ for arbitrary $x \in X$. Denote the collection of idempotents in M by J. Then $uM = \{up \mid p \in M\}$ is a subgroup of M for every $u \in J$ and $\{vM \mid v \in J\}$ is a partition of M. Let $u \in J$ be a fixed idempotent and put $G = uM$. Then uM can be provided with a nice compact T_1 topology, the τ-topology, such that the left and right translations and the inverse are τ-homeomorphisms of uM. For every minimal ttg with u-invariant base point x_0 we may consider the stabilizer of X with respect to x_0 in G:

$\mathcal{G}(X,x_0) = \{\alpha \in G \mid \alpha x_0 = x_0\}$. Then $\mathcal{G}(X,x_0)$ is a τ-closed subgroup of G, the so called *Ellis group* of X with respect to x_0. Moreover, every τ-closed subgroup F of G is an Ellis group of some minimal ttg. These Ellis groups are of great importance in the structure theory of minimal ttg's. Another strong device is the notion of a quasi-factor. Every ttg (T,X) induces a hypertransformation group $(T,2^X)$, where 2^X denotes the compact T_2 space of all nonvacuous closed subsets of X. The action of T on 2^X is defined by $(t,A) \mapsto \{ta \mid a \in A\}$ for every $t \in T$ and $A \in 2^X$, and it is continuous (cf. [6]). Every homomorphism ϕ: $X \to Y$ induces a homomorphism $\phi^*: 2^X \to 2^Y$, defined by $\phi^*(A) = \phi[A]$ for every $A \in 2^X$, and if ϕ is surjective, also a homomorphism $\phi^{\leftarrow *}: 2^Y \to 2^X$, defined by $\phi^{\leftarrow *}(A) = \phi^{\leftarrow}[A]$ for every $A \in 2^Y$. If ϕ is open and surjective, it is easy to see that $\phi^{\leftarrow *}|_Y$ is a topological embedding.

There are two possible actions of M on 2^X, for $p \in M$ and $A \in 2^X$ may stand for $\{pa \mid a \in A\}$, or for pA in the orbit closure of A in 2^X; and, in general, both sets are different. To avoid this ambivalence, we shall denote the last version by $p \circ A$ and it can be proved that $p \circ A = \{y \in X \mid$ there is a net $\{t_i\} \to p$ in T and there are $a_i \in A$ with $\{t_i a_i\} \to y\}$.

A quasi-factor of X is defined as a minimal sub-ttg of 2^X and it is easy to see that every quasi-factor $\mathcal{X}$ of X has the form $\mathcal{X} = QF(A,X) = \{p \circ A \mid p \in M\}$ for some almost periodic $A \in 2^X$. If X is minimal, then $X \cong QF(x,X)$ for every $x \in X$.

Quasi-factors are very useful in problems concerning disjointness of minimal ttg's. Two minimal ttg's are called *disjoint*, if their product ttg is again minimal. If K is a collection of minimal ttg's, we denote by $K^{\perp}$ the collection of all minimal ttg's, which are disjoint from every member of K. For instance, $\mathcal{D}^{\perp}$ $(\mathcal{P}^{\perp})$ is the collection of all minimal ttg's disjoint from every distal (proximal) ttg or, equivalently, disjoint from the universal disjoint (proximal) ttg D (P) for T.

1. PREREQUISITES

In this section we shall be concerned with some basic properties of disjointness, quasi-factors and extensions. Most results are stated without proof, since the proofs can be found in [1] and [10]. In the sequel $\phi: X \to Y$ denotes a homomorphism of minimal ttg's.

1.1. THEOREM.

1° *For every quasi-factor* $\mathfrak{X}$ *of* X, $\phi^*[\mathfrak{X}]$ *is a quasi-factor of Y, and* $\phi^*[\mathfrak{X}]$ *is trivial iff* $\phi[A] = Y$ *for some* $A \in \mathfrak{X}$.

2° *If* $\mathfrak{X}$ *is a quasi-factor of* X *with* $\phi^*[\mathfrak{X}]$ *nontrivial, then* $\mathfrak{X} \not\perp Y$.

3° *If* $\mathfrak{Y}$ *is a nontrivial quasi-factor of* Y, *then* $X \not\perp \mathfrak{Y}$.

4° *If* ϕ *is open, then every quasi-factor of* Y *is a quasi-factor of* X.

5° *If* ϕ *is open, then* $\mathfrak{X} \not\perp Y$ *for every nontrivial quasi-factor* $\mathfrak{X}$ *of* X, *such that there is an* $A \in \mathfrak{X}$ *with* $\phi[X \backslash A] \neq Y$.

1.2. THEOREM. *Let* ϕ *be distal. Then every nontrivial quasi-factor* $\mathfrak{X}$ *of* X *with* $\mathfrak{X} \perp Y$ *is distal.*

1.3. COROLLARY. *Let* ϕ *be distal and* $X \in \mathcal{D}^\perp$. *Then* $\mathfrak{X} \not\perp Y$ *for every nontrivial quasi-factor* $\mathfrak{X}$ *of* X.

In order to prove a similar statement with ϕ proximal, we need the following theorem of VEECH ([9], remark on p.814). But first we need a definition. Let $x_0 = ux_0 \in X$ and $F = \mathfrak{G}(Y,\phi(x_0))$. Then ϕ is called a RIC *extension* if $\overleftarrow{\phi}(p\phi(x_0)) = p \circ F x_0$ for all $p \in M$.

1.4. THEOREM. *Let* ϕ *be a* RIC *extension and let* $\psi: Z \to Y$ *be a homomorphism of minimal ttg's. Then* $R_{\phi\psi} = \{(x,z) \in X \times Z \mid \phi(x) = \psi(z)\}$ *has a dense set of almost periodic points.*

1.5. THEOREM. *Let* ϕ *be proximal and* $X \in \mathcal{P}^\perp$. *Then* $\mathfrak{X} \not\perp Y$ *for every nontrivial quasi-factor* $\mathfrak{X}$ *of* X.

PROOF. Suppose that $\mathfrak{X} \perp Y$ for some nontrivial quasi-factor $\mathfrak{X}$ of X. Since $\mathfrak{X} \times Y$ is minimal and $1_{\mathfrak{X}} \times \phi: \mathfrak{X} \times X \to \mathfrak{X} \times Y$ is proximal, it follows that $\mathfrak{X} \times X$ has a unique minimal subset. But $\psi: X \to \{*\}$ is a RIC extension ([5],X.1.3), so

1.4 implies that $\mathcal{X}\times X$ has a dense subset of almost periodic points. Since $\mathcal{X}\times X$ is minimal and $\mathcal{X}$ is nontrivial, this violates $1.1.3^{\circ}$. □

An important type of extension related to disjointness is the highly proximal extension ([1]). A homomorphism $\phi: X \to Y$ of minimal ttg's is called *highly proximal* (h.p. for short), if ϕ satisfies one of the following four equivalent properties:

a For some $y \in Y$ there is a net $\{t_n\}_n$ in T, such that the net $\{t_n\phi^{\leftarrow}(y)\}_n$ tends to a singleton in 2^X.

b Every nonempty open subset of X contains a fiber $\phi^{\leftarrow}(y)$ for some $y \in Y$.

c $\phi[A] \neq Y$ for every closed $A \subseteq X$ with $A \neq X$.

d If $y \in Y$, $x \in \phi^{\leftarrow}(y)$ and $p \in M$, then $p\circ\phi^{\leftarrow}(y) = \{px\}$.

Define two minimal ttg's to be h.p. equivalent, if they have a common h.p. extension. Then there is a partition of the collection of minimal ttg's in h.p. equivalence classes. This partition is compatible with the notion of disjointness. That means, if $(X_1 \sim X_2)_{h.p.}$ and $(Y_1 \sim Y_2)_{h.p.}$, then $X_1 \perp Y_1$ iff $X_2 \perp Y_2$ ([1] Thm.1.2). Every equivalence class has a unique maximal element: the maximally highly proximal extension of each of its members. If $\gamma: M \to X$ is an extension and $x \in X$, then $X^* = \mathrm{QF}(u\circ\gamma^{\leftarrow}(x), M)$ is the maximal h.p. extension of each member in the equivalence class of X, and X^* is independent of the choice of $x \in X$. We shall call such a ttg X^* *maximally highly proximal*.

1.6. THEOREM.

1° Y *is maximally highly proximal iff every* $\phi: X \to Y$ *is open.*

2° *If* Y *is a factor of* X *then* Y^* *is a factor of* X^*.

3° $X \perp Y$ *iff* $X^* \perp Y^*$.

4° $X \not\perp Y$ *iff* Y *has a nontrivial quasi-factor which is a factor of* X^*.

In the study of disjointness classes the following closure operator on collections of minimal ttg's is natural, since disjointness is preserved under h.p. extensions and factors. Define for a collection K of minimal ttg's the closure $[K]$ of K to be

$$[K] = \{Z \mid Z \text{ is a factor of } Y^* \text{ for some } Y \in K\}.$$

Clearly, this defines a closure operator and $[K]^{\perp} = K^{\perp} = [K^{\perp}]$.

1.7. THEOREM. *Let* K *be a collection of minimal ttg's. Then* $X \in K^{\perp}$ *iff* $\mathcal{X} \notin [K]$ *for every nontrivial quasi-factor* $\mathcal{X}$ *of* X.

1.8. EXAMPLES.

1° Let D (P) be the universal minimal distal (proximal) ttg for T. Then
$[\mathcal{D}] = [\{D\}] = \{Z \mid Z \text{ is a factor of } D^{*}\}$ and
$[\mathcal{P}] = [\{P\}] = \{Z \mid Z \text{ is a factor of } P^{*} = P\} = \mathcal{P}$.

2° Let F be a τ-closed subgroup of G. Define $M(F) = \{Y \mid Y$ is a minimal and $\mathcal{G}(Y,y_0) \supseteq F$ for some $y_0 \in Y\}$. Then $M(F) = [M(F)] = [\{QF(u{\circ}F,M)\}]$.
Remark that $M(G) = \mathcal{P}$.

2. WEAK MIXING AND DISJOINTNESS

First we recall some definitions.

A ttg X is called *ergodic* if X does not contain proper closed invariant subsets with nonempty interior. Two ttg's X and Y are called *weakly disjoint* ($X \doteq Y$) if $X \times Y$ is ergodic, and X is *weakly mixing* if $X \doteq X$ (for instance, every proximal minimal ttg is weakly mixing ([5],II.2.2)). Define $L^{\doteq}$: $\{X \mid X$ is minimal and $X \doteq Y$ for every $Y \in L\}$ for a collection L of minimal ttg's. A homomorphism $\phi: X \to Y$ is called weakly mixing if $R_{\phi} = \{(x_1,x_2) \mid \phi(x_1) = \phi(x_2)\} \subseteq X \times Y$ is ergodic. Denote by $\mathcal{WM}$ the collection of weakly mixing minimal ttg's. Let X' and Y' be factors of X and Y, respectively, and let $X \doteq Y$. Since ergodicity is preserved under factors, it follows that $X' \doteq Y'$. In particular, $\mathcal{WM}$ is closed under factors.

2.1. THEOREM. *Let* X *and* Y *be minimal and let* $\phi: X' \to X$ *and* $\psi: Y' \to Y$ *be h.p. extensions. Then* $X \doteq Y$ *iff* $X' \doteq Y'$.

PROOF. Assume that $X' \times Y'$ is not ergodic. Then there exists a closed invariant subset A of $X' \times Y'$ with nonempty interior and $A \neq X'{\times}Y'$. Also $B = \overline{X'{\times}Y' \setminus A}$ is a closed invariant subset of $X' \times Y'$ with nonempty interior and $B \neq X'{\times}Y'$. Obviously, $X{\times}Y = \phi{\times}\psi[A] \cup \phi{\times}\psi[B]$. Choose open subsets U and V of X' and Y' with $U{\times}V \subseteq \overset{\circ}{A}$. Since ϕ and ψ are h.p., there are $x \in X$ and $y \in Y$ such that $\overleftarrow{\phi}(x) \subseteq U$ and $\overleftarrow{\psi}(y) \subseteq V$, so $\overleftarrow{(\phi{\times}\psi)}(x,y) \subseteq U{\times}V \subseteq \overset{\circ}{A}$.

Consequently, $(x,y) \notin \phi\times\psi[B]$. Similarly, also $\phi\times\psi[A] \neq X\times Y$. Since $\phi\times\psi[A]$ and $\phi\times\psi[B]$ are closed and invariant, it follows from the ergodicity of $X\times Y$ that they have empty interiors; contradiction! □

2.2. COROLLARY.
a $[K]^{\dot{-}} = [K^{\dot{-}}] = K^{\dot{-}}$.
b $[\mathcal{WM}] = \mathcal{WM}$.
c $X \in \mathcal{WM}^{\perp}$ iff X has no weakly mixing quasi-factors.

PROOF. In view of 2.1, a and b are obvious; c follows from Theorem 1.7 and b. □

The following theorem of ELLIS [3] shows the hostility between the properties weak mixing and distal.

2.3. THEOREM. *A distal and ergodic ttg is minimal.*

2.4. THEOREM. *Let* $Y \in \mathcal{D}^{\perp}$ *and let* $\phi: X \to Y$ *be a weakly mixing homomorphism of minimal ttg's. Then* $X \in \mathcal{D}^{\perp}$.

PROOF. Suppose that $X \notin \mathcal{D}^{\perp}$. Then there exists a homomorphism $\phi: X \to Z$ with Z nontrivial and distal. Since R_ϕ is ergodic, it follows that $\psi\times\psi(R_\phi) \subseteq Z\times Z$ is ergodic and distal, hence by 3.3 minimal. Since $\Delta_X \subseteq R_\phi$ and so $\psi\times\psi(R_\phi) \supseteq \Delta_Z$, it follows that $\psi\times\psi(R_\phi) = \Delta_Z$ and also $R_\phi \subseteq R_\psi = (\psi\times\psi)^{\leftarrow}[\Delta_Z]$. But then $Z = X/R_\psi$ is a factor of $Y = X/R_\phi$, but Z is distal and $Y \in \mathcal{D}^{\perp}$; contradiction. □

2.5. COROLLARY. $\mathcal{WM} \subseteq \mathcal{D}^{\perp}$.

The following characterization of the incontractible, weakly mixing minimal ttg's is an easy consequence of ([9], 2.1.6) and can be found in ([10],4.3). So we state it here without proof.

2.6. THEOREM. $\mathcal{P}^{\perp} \cap \mathcal{WM} = \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp}$.

3. TOWERING INFERNO

A fruitful approach in determining the structure of minimal ttg's is the concept of a tower, i.e., a succession of extensions. We shall define certain collections of minimal ttg's which are in a sense built up by specific types of towers, following [4], but somewhat different from [7] and [9].

Let ν be an ordinal and let $\phi: X \to Y$ be a homomorphism of minimal ttg's. A *tower of height* ν *for* ϕ is defined to be a system $\{(W_\alpha,\omega_\alpha) \mid \alpha \le \nu\}$ of minimal ttg's with u-invariant base points, such that

1° $W_0 = Y$ and $W_\nu = X$;

2° for every $\alpha < \nu$, (W_α,ω_α) is a factor of $(W_{\alpha+1},\omega_{\alpha+1})$ under a homomorphism ϕ_α with $\phi_\alpha(\omega_{\alpha+1}) = \omega_\alpha$;

3° if α is a limit ordinal, then $(W_\alpha,\omega_\alpha) = \bigvee\{(W_\beta,\omega_\beta) \mid \beta < \alpha\}$;

4° ϕ is the inverse limit of $\{\phi_\alpha \mid \alpha < \nu,\ \alpha$ non limit ordinal$\}$.

Here $\bigvee\{(W_\beta,\omega_\beta) \mid \beta < \alpha\}$ denotes the minimal orbit closure of $(\phi_\beta(\omega_{\beta+1}))_{\beta<\alpha}$ in $\Pi\{W_\beta \mid \beta < \alpha\}$.

The homomorphism $\phi: X \to Y$ of minimal ttg's will be called a *strictly*-I (*strictly*-PI) (*strictly*-HPI) (*strictly*-PD) (*strictly*-HPD) extension, if a tower for ϕ exists such that every ϕ_α is almost periodic (proximal or almost periodic) (highly proximal or almost periodic) (proximal or distal) (highly proximal or distal). Let B be one of the symbols I, PI, HPI, PD or HPD. homomorphism $\phi: X \to Y$ of minimal ttg's will be called a B *extension* if there exist a minimal ttg Z and homomorphisms $\psi: Z \to Y$ and $\theta: Z \to X$ such that $\phi\circ\theta = \psi$ and ψ is a strictly-B extension. A minimal ttg X is called (strictly-)B, if $\phi: X \to \{*\}$ is a (strictly-)B extension.

3.1. Denote by $\mathcal{B}$ the collection of minimal B ttg's. Then the following relations hold:

<u>a</u> $\mathcal{D} = \mathcal{I}$ (ELLIS [3]),

<u>b</u> $\mathcal{I} \subseteq \mathcal{HPI} \subseteq \mathcal{HPD} \subseteq \mathcal{PD}$,

<u>c</u> $\mathcal{HPI} \subseteq \mathcal{PI} \subseteq \mathcal{PD}$,

<u>d</u> $\mathcal{HPI} = \mathcal{HPD}$ and $\mathcal{PI} = \mathcal{PD}$ for metric minimal ttg's, or if T is locally compact and σ-compact.

Note, that our definitions of PI- and HPI ttg's coincide with those in [1],[4] and [5] (see [5], corollary X.4.3 and [1] corollary III.1). The definitions of PD ttg and PD extension in [7] and [8] and those of I-, PI- and HPI extension in [9] are just our definitions with prefix strictly.

3.2. REMARK. If $\mathcal{B}$ is one of the classes $\mathcal{PI}$, $\mathcal{PD}$, $\mathcal{HPI}$ or $\mathcal{HPD}$, then $[\mathcal{B}] = \mathcal{B}$. For let X be a B ttg, then there exists a strictly-B ttg Z, which has X as a factor. Obviously, every factor of X is a factor of Z and so a B ttg. Let X' be an h.p. extension of X. Then X' is a factor of X^*. Since X^* is a factor of Z^* and Z^* is a strictly-B ttg, it follows that X' as a factor of Z^* is a B-ttg too.

Let F be a τ-closed subgroup of $\mathcal{G}$. Then $H(F)$ is defined to be the smallest τ-closed normal subgroup of F, such that $F/H(F)$ is a compact T_2 topological group ([5],IX.1.9). Set $H_1(F) := H(F)$, $H_{\alpha+1}(F) := H(H_\alpha(F))$ for every ordinal α and for a limit ordinal α, $H_\alpha(F) := \cap\{H_\beta(F) \mid \beta < \alpha\}$. Define F_∞ to be the limit of $\{H_\alpha(F)\}_\alpha$.

For the sequel we have to recall the next two lemmas:

3.3. LEMMA ([5],IX.2.1(4)). *Let* ϕ: X $\to$ Y *be a distal homomorphism of minimal ttg's, and let* H *and* F *be the* ϕ*-compatible Ellis groups of* X *and* Y, *respectively (i.e., the Ellis groups with respect to* u-*invariant base points* x_0 *and* $\phi(x_0)$*). Then* ϕ *is almost periodic iff* $H(F) \subseteq H$.

3.4. LEMMA ([5],X.2.1). *Let* ϕ: X $\to$ Y *be a RIC extension of minimal ttg's, and let* H *and* F *be the* ϕ*-compatible Ellis groups of* X *and* Y, *respectively. Then there exist a minimal ttg* Z, *with Ellis group* $H(F)H$, *and homomorphisms* ψ: X $\to$ Z *and* θ: Z $\to$ Y, *such that* θ *is almost periodic and* $\theta\circ\psi = \phi$.

3.5. PROPOSITION. *Let* K *be a* τ*-closed subgroup of* $\mathcal{G}$. *Then the canonical map* η: $QF(u\circ H(K),M) \to QF(u\circ K,M)$ *is a strictly*-PI *extension, and so* ξ: $QF(u\circ K_\infty,M) \to QF(u\circ K,M)$ *is strictly*-PI.

PROOF. Since η is a RIC extension we may find a minimal ttg Z with Ellis group $H(K)\cdot H(K) = H(K)$ and homomorphisms ψ and θ with θ almost periodic and $\eta = \theta\circ\psi$. Since ψ: $QF(u\circ H(K),M) \to Z$ is proximal, it follows that η is strictly-PI.

3.6. <u>THEOREM</u>. *Let* ϕ: X $\to$ Y *be a homomorphism of minimal ttg's and let* H *and* F *be the Ellis groups of* X *and* Y, *respectively. Then* ϕ *is a* PI *extension iff* $F_\infty \subseteq H$.

<u>PROOF</u>. " $\Leftarrow$ ". Let κ: $QF(u\circ F,M) \to Y$ be the canonical proximal homomorphism. Since ψ: $QF(u\circ F_\infty,M) \to QF(u\circ F,M)$ is strictly PI, so is $\kappa\circ\psi$. But $F_\infty \subseteq H$, so X is a factor of $QF(u\circ F_\infty,M)$ and ϕ is a PI extension. (Note that $u\circ F_\infty \overset{\psi}{\longmapsto} u\circ F \overset{\kappa}{\longmapsto} \phi(x_0)$ and $u\circ F_\infty \longmapsto u\circ H \longmapsto x_0$.)

" $\Rightarrow$ ". We only have to prove that for a strictly-PI extension ϕ: X $\to$ Y we have $F_\infty \subseteq H$. Let ϕ be strictly-PI; then for every $(W_{\alpha+1},\omega_{\alpha+1})$ in the tower for ϕ we have, by 3.3, $\mathcal{G}(W_{\alpha+1},\omega_{\alpha+1}) \supseteq H(\mathcal{G}(W_\alpha,\omega_\alpha))$. Now it follows easily that $H = \mathcal{G}(W_\nu,\omega_\nu) \supseteq F_\infty = \mathcal{G}(W_0,\omega_0)_\infty$. □

3.7. <u>COROLLARY</u>.

1° $\mathcal{PI} = \mathcal{M}(G_\infty) = [\{QF(u\circ G_\infty,M)\}]$.

2° *Let* X *be minimal with Ellis group* H. *Then* $X \in \mathcal{PI}^\perp$ *iff* $u\circ G_\infty x = X$ *for some* $x \in X$ ([10],2.12.d').

Note that for τ-closed subgroups K and L of G with $K \subseteq L$, we have that $H(K) \subseteq H(L) \cap K \subseteq H(L)$ and so $K_\infty \subseteq L_\infty$. Now it follows that, with notation as in 3.6, ϕ is a PI extension iff $F_\infty = H_\infty$. Also it is clear that, if $X \in \mathcal{PI}$, then ϕ is a PI extension.

Whether or not every distal homomorphism ϕ: X $\to$ Y of minimal ttg's without countability assumptions is strictly-I is still an open question for nontrivial Y. The following theorem gives a necessary and sufficient condition for a distal homomorphism to be strictly-I. First note that if ϕ is a homomorphism of minimal ttg's with $\phi = \psi\circ\theta$, then ϕ is distal iff ψ and θ are distal.

3.8. <u>THEOREM</u>. *Let* ϕ: X $\to$ Y *be a distal homomorphism of minimal ttg's. Then the following are equivalent:*

1° ϕ *is a* PI *extension;*

2° ϕ *is an* HPI *extension;*

3° ϕ *is a strictly-*I *extension.*

<u>PROOF</u>. The implications 3° $\Rightarrow$ 2° $\Rightarrow$ 1° are obvious.

$1^\circ \Rightarrow 3^\circ$. Let H and F be the Ellis groups of X and Y, respectively. Set $Z_0 := Y$ and $\psi_0 := \phi$. Then $\psi_0\colon X \to Z_0$ is distal and so RIC. By 3.4 we may find a minimal ttg Z_1 with Ellis group $H(F)H$ and homomorphisms ψ_1 and θ_1 with θ_1 almost periodic, such that $\psi_0 = \theta_1 \circ \psi_1$. Since ψ_0 is distal, so is ψ_1 and we may repeat the procedure for $\psi_1\colon X \to Z_1$. We then find a minimal ttg Z_2 with Ellis group $H(H(F)\cdot H)\cdot H = H_2(F)H$ ([5],X.4.1) and homomorphisms $\psi_2\colon X \to Z_2$, $\theta_2\colon Z_2 \to Z_1$ with θ_2 almost periodic, so $\psi_0 = \theta_1 \circ \theta_2 \circ \psi_2$. Transfinite induction gives us a minimal ttg Z_∞ and homomorphisms $\psi_\infty\colon X \to Z_\infty$ and $\theta_\infty = \theta_1 \circ \theta_2 \circ \ldots\colon Z_\infty \to Z_0$, such that the Ellis group of Z_∞ equals $F_\infty H = H$ and $\phi = \psi_0 = \theta_\infty \circ \psi_\infty$. Since ψ_∞ is distal and proximal $\psi_\infty = id_X$ and $\phi = \theta_\infty$, which is strictly-I. □

3.9. <u>COROLLARY</u>. *If* $X \in$ *PI and* $\phi\colon X \to Y$ *is distal, then* ϕ *is strictly*-I.

In case $\phi\colon X \to Y$ is distal but not PI, the last step in the proof of 3.8 fails, but we can say something about it. For that purpose we need the following theorem, which is in fact a special case of [9], 2.6.3.

3.10. <u>THEOREM</u>. *Let* $\phi\colon X \to Y$ *be a homomorphism of minimal ttg's, such that no nontrivial almost periodic extension of* Y *between* X *and* Y *exists (i.e., if* $\phi = \theta \circ \psi$ *with* θ *almost periodic, then* $\theta = id_Y$*), and* R_ϕ *has a dense set of almost periodic points. Then* ϕ *is weakly mixing.*

3.11. <u>LEMMA</u>. *Let* $\phi\colon X \to Y$ *be a RIC extension and let* H, F *and* K *be* τ*-closed subgroups of* G *with* $H \subseteq K \subseteq F_\infty H$*, such that* $\mathfrak{G}(X,x_0) = K$ *and* $\mathfrak{G}(Y,\phi(x_0)) = F_\infty H$. *Then* ϕ *is weakly mixing. In particular, for* $H = \mathfrak{G}(M,u)$ *this implies that every extension of the universal* PI *extension of ttg's with Ellis group* F *is weakly mixing.*

<u>PROOF</u>. By 3.4 we may find a minimal ttg Z with Ellis group $H(F_\infty H)K$, and homomorphisms ψ and θ such that $\theta\colon Z \to Y$ is almost periodic and $\phi = \theta \circ \psi$. Since $H(F_\infty H)K \subseteq F_\infty H = H(F_\infty)H = H(F_\infty H)H \subseteq H(F_\infty H)K$ and so $F_\infty H = H(F_\infty H)K$, it follows that θ is proximal and consequently $\theta = id_Y$. By 1.4 and 3.10 ϕ turns out to be weakly mixing. □

3.12. <u>COROLLARY</u>. *Let* $\phi\colon X \to Y$ *be distal and not* PI. *Then* $\phi = \theta \circ \psi$ *with* θ *a strictly*-I *extension and* ψ *distal and weakly mixing.*

PROOF. Following the construction in the proof of 3.8, the last step gives us $\psi_\infty: X \to Z_\infty$. The Ellis group of Z_∞ is $F_\infty H$, but $F_\infty H \neq H$ since $F_\infty \not\subseteq H$. Now 3.11 applies, knowing that the distal extension ψ_∞ also is a RIC extension.□

Note, that with the same type of construction we may prove that the RIC extension ϕ: $QF(u \circ H, M) \to QF(u \circ F, M)$, with H and F τ-closed subgroups of G, can be written as $\phi = \theta \circ \psi$, where θ: $QF(u \circ F_\infty H, M) \to QF(u \circ F, M)$ is strictly-PI and θ: $QF(u \circ H, M) \to QF(u \circ F_\infty H, M)$ is weakly mixing.

4. DISJOINTNESS CLASSES

We now want to give some relations between collections of minimal ttg's, leading to the diagrams on pages 14 and 15. These relations are far from complete since we did not investigate towers with occurrences of weakly mixing extensions. Note that every minimal ttg is a factor of a weakly mixing extension of the universal PI ttg. For let X be minimal with Ellis group H; then X is a factor of $QF(u \circ H_\infty, M)$ which is, by 3.11, a weakly mixing extension of $QF(u \circ G_\infty, M)$. Dealing with disjointness, the following theorem ([10],3.4) is useful.

4.1. THEOREM. *Let* $X \in \mathcal{P}^\perp$ *and* Y *minimal. Let* H *and* F *(in* G*) be Ellis groups of* X *and* Y*, respectively. Then* $X \perp Y$ *iff* $HF = G$.

4.2. THEOREM. *Let* $\mathcal{K}$ *be a collection of minimal ttg's.*
a *If* $\mathcal{K} \subseteq \mathcal{D}^\perp$ *then* $\mathcal{K}^\perp$ *is closed under distal extensions.*
b *If* $\mathcal{K} \subseteq \mathcal{P}^\perp$ *then* $\mathcal{K}^\perp$ *is closed under proximal extensions.*
c *If* $\mathcal{K} \subseteq \mathcal{P}^\perp \cap \mathcal{D}^\perp$ *then* $\mathcal{K}^\perp$ *is closed under* PD *extensions.*

PROOF. a. Let $Y \in \mathcal{K}^\perp$ *and let* ϕ: $X \to Y$ *be distal. Suppose that* $X \notin \mathcal{K}^\perp$; then there exists a $Z \in \mathcal{K}$ with $X \not\perp Z$. By 1.6.4° there is a nontrivial quasi-factor $\mathcal{X}$ of X which is a factor of Z^*. Since $Z^* \perp Y$, it follows that $\mathcal{X} \perp Y$ and so, by 1.2, $\mathcal{X}$ is distal. But $Z \in \mathcal{K} \subseteq \mathcal{D}^\perp$, hence $Z^* \in \mathcal{D}^\perp$. So Z^* does not admit nontrivial distal factors. A contradiction.

b. Let $Y \in \mathcal{K}^\perp$. Let ϕ: $X \to Y$ be proximal and let H and F be the Ellis groups of X and Y. Suppose that $X \notin \mathcal{K}^\perp$; then there is a $Z \in \mathcal{K}$ with $X \not\perp Z$, while

$Y \perp Z$. Since $Z \in \mathcal{K} \subseteq \mathcal{P}^{\perp}$, it follows from 4.1 that $HK \neq G$ and $FK = G$, where K denotes the Ellis group of Z. However, by the proximality of ϕ we have $H = F$.

c. From a and b it is clear that $\mathcal{K}^{\perp}$ is closed under strictly-PD extensions. Since $\mathcal{K}^{\perp}$ is closed under factors, the theorem follows. □

4.3. EXAMPLES.

1° $\mathcal{P}^{\perp}$, $\mathcal{WM}^{\perp}$, $\mathcal{D}^{\perp\perp}$ and $\mathcal{PD}^{\perp\perp}$ are closed under distal extensions and so under HPD extensions.

2° $\mathcal{D}^{\perp}$, $\mathcal{WM}^{\perp\perp}$, $\mathcal{P}^{\perp\perp}$ and $\mathcal{PD}^{\perp\perp}$ are closed under proximal extensions.

3° $\mathcal{PD}^{\perp\perp}$ is closed under PD extensions.

Let F be a τ-closed subgroup of G and let D be the Ellis group (in G) of the universal minimal distal ttg D.

4° $M(F)^{\perp\perp}$ is closed under proximal extensions.

5° If $F \subseteq D$ then $M(F)^{\perp\perp}$ is closed under distal extensions, hence under PD extensions.

6° If $FD = G$ then $M(F)^{\perp}$ is closed under distal extensions and so under HPD extensions.

4.4. COROLLARY.

a $\mathcal{HPD} \subseteq \cap\{M(F)^{\perp} \mid F \subseteq G$ τ-closed subgroup with $FD = G\} \subseteq \mathcal{D}^{\perp\perp} \subseteq \mathcal{WM}^{\perp} \subseteq \mathcal{P}^{\perp}$.

b $\mathcal{HPD}^{\perp} = \mathcal{HPI}^{\perp} = \mathcal{I}^{\perp} = \mathcal{D}^{\perp}$.

c $M(F_{\infty})^{\perp} = M(F)^{\perp}$ *if* $F \subseteq D$.

d $M(G_{\infty}) = \mathcal{PI} \subseteq \mathcal{PD} \subseteq M(F)^{\perp\perp}$ *for every* τ*-closed subgroup* $F \subseteq D$.

e $M(G_{\infty})^{\perp} = \mathcal{PI}^{\perp} = \mathcal{PD}^{\perp} = M(F)^{\perp} \subseteq \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp}$ *for every* τ*-closed subgroup* F *of* G *with* $G_{\infty} \subseteq F \subseteq D$.

PROOF. a. Note, that $\mathcal{D}^{\perp} \subseteq \cup\{M(F) \mid F \subseteq G$ τ-closed subgroup with $FD = G\}$. The other inclusions follow from 4.3.6°, 2.5 and ([5],II.2.2).

b. Obviously, $\mathcal{HPD}^{\perp} \subseteq \mathcal{HPI}^{\perp} \subseteq \mathcal{I}^{\perp} \subseteq \mathcal{D}^{\perp}$. From a it follows that $\mathcal{D}^{\perp} \subseteq \mathcal{HPD}^{\perp}$.

c. By 3.5 $QF(u\circ F_{\infty},M) \to QF(u\circ F,M)$ is a PI extension, so by 4.3.5° $M(F_{\infty}) \subseteq M(F)^{\perp\perp}$, hence $M(F) \subseteq M(F_{\infty}) \subseteq M(F)^{\perp\perp}$ and $M(F)^{\perp} = M(F_{\infty})^{\perp}$.

d. Clear by 3.5.

e. From d it is clear that $M(G_{\infty})^{\perp\perp} = \mathcal{PI}^{\perp\perp} \subseteq \mathcal{PD}^{\perp\perp} \subseteq M(F)^{\perp\perp}$. Since $G_{\infty} \subseteq F$

and consequently $M(F) \subseteq M(G_\infty)$, it follows that $M(F)^{\perp\perp} = M(G_\infty)^{\perp\perp} = \mathcal{PI}^{\perp\perp} = \mathcal{PD}^{\perp\perp}$ and so $M(F)^\perp = M(G_\infty)^\perp = \mathcal{PI}^\perp = \mathcal{PD}^\perp$. □

4.5. THEOREM. *Let* K *be a collection of minimal ttg's.*

a $K^\perp$ *is closed under distal extensions within* $\mathcal{D}^\perp$. *(I.e., let* $Y \in K^\perp$ *and let* $\phi: X \to Y$ *be distal. If* $X \in \mathcal{D}^\perp$ *then* $X \in K^\perp$.*)*

b $K^\perp$ *is closed under proximal extensions within* $\mathcal{P}^\perp$.

c $K^\perp$ *is closed under* PD *extensions within* $\mathcal{P}^\perp \cap \mathcal{D}^\perp$.

PROOF. a (b). Let $\phi: X \to Y$ be a distal (proximal) extension with $X \in \mathcal{D}^\perp$ ($\mathcal{P}^\perp$), $Y \in K^\perp$ and suppose $X \notin K^\perp$. Then there exists, by 1.7, a nontrivial quasi-factor $\mathcal{X}$ of X with $\mathcal{X} \in [K]$. Now $\mathcal{X} \perp Y$, which contradicts 1.3 (1.5).

c. Follows from a and b. □

4.6. EXAMPLES.

1° $\mathcal{P}^{\perp\perp}$ and $\mathcal{WM}^{\perp\perp}$ are closed under distal extensions within $\mathcal{D}^\perp$; hence, by example 4.3.2°, closed under PD extensions within $\mathcal{D}^\perp$.

2° $\mathcal{D}^{\perp\perp}$ and $\mathcal{WM}^\perp$ are closed under proximal extension within $\mathcal{P}^\perp$; hence, by example 4.3.1°, closed under PD extensions within $\mathcal{P}^\perp$.

4.7. COROLLARY.

a $\mathcal{D}^\perp \cap \mathcal{PD} = \mathcal{P}^{\perp\perp} \cap \mathcal{PD} = \mathcal{WM}^{\perp\perp} \cap \mathcal{PD}$.

b $\mathcal{P}^\perp \cap \mathcal{PD} = \mathcal{D}^{\perp\perp} \cap \mathcal{PD} = \mathcal{WM}^\perp \cap \mathcal{PD}$.

PROOF. a. From 4.6.1° it follows that $\mathcal{D}^\perp \cap \mathcal{PD} \subseteq \mathcal{P}^{\perp\perp}$, but $\mathcal{P}^{\perp\perp} \subseteq \mathcal{WM}^{\perp\perp} \subseteq \mathcal{D}^\perp$.

b. From 4.6.2° it follows that $\mathcal{P}^\perp \cap \mathcal{PD} \subseteq \mathcal{D}^{\perp\perp}$, but $\mathcal{D}^{\perp\perp} \subseteq \mathcal{WM}^\perp \subseteq \mathcal{P}^\perp$. □

4.8. THEOREM. $\mathcal{PD}^\perp = \mathcal{PI}^\perp = \mathcal{P}^\perp \cap \mathcal{D}^\perp$.

PROOF. Since in the proofs of [10], 5.3 and 5.4 only the distality of almost periodic extensions was used, we may replace $\mathcal{PI}$ by $\mathcal{PD}$ in those theorems. □

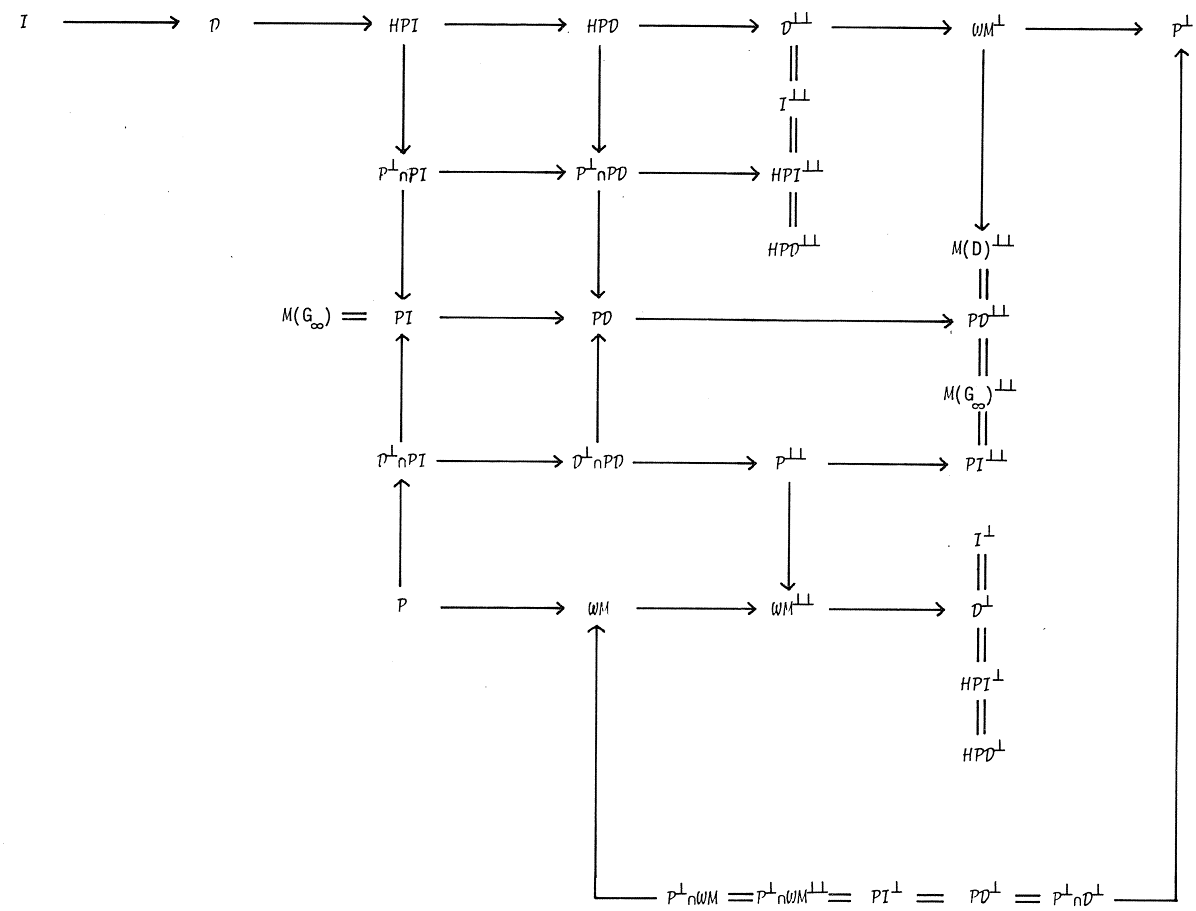
4.9. COROLLARY.

a $\mathcal{PD}^\perp = \mathcal{P}^\perp \cap \mathcal{D}^\perp = \mathcal{P}^\perp \cap \mathcal{WM} = \mathcal{P}^\perp \cap \mathcal{WM}^{\perp\perp}$, *so* $\mathcal{PD}^\perp \subseteq \mathcal{WM}$ *and* $\mathcal{WM}^\perp \subseteq \mathcal{PD}^{\perp\perp}$.

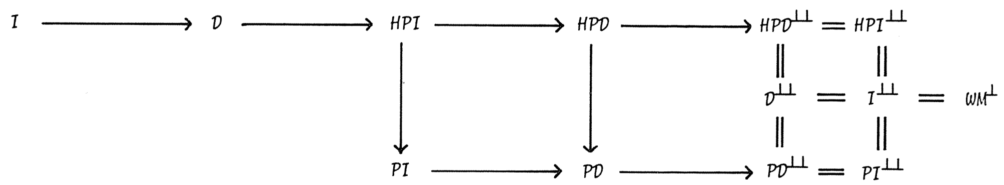
b *If* T *is strongly amenable, we have* $\mathcal{PD}^\perp = \mathcal{D}^\perp = \mathcal{WM} = \mathcal{WM}^{\perp\perp}$.

(T is called *strongly amenable* iff the universal proximal minimal ttg for T is trivial.)

ARBITRAIRY PHASEGROUP



STRONGLY AMENABLE PHASEGROUP



$$I^{\perp} = D^{\perp} = HPI^{\perp} = HPD^{\perp} = PI^{\perp} = PD^{\perp} = WM = WM^{\perp\perp}$$

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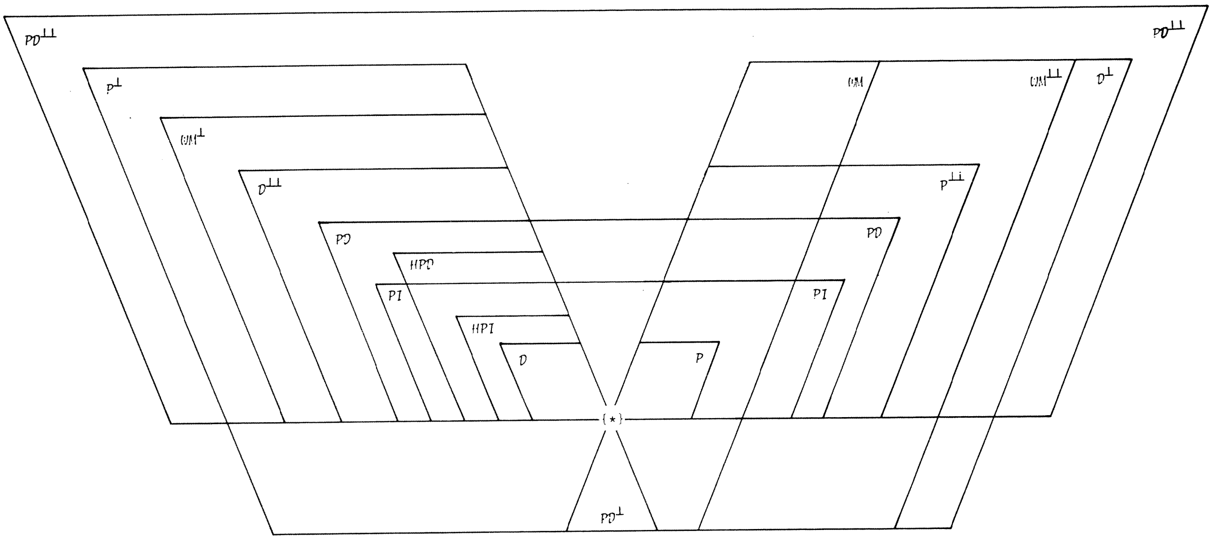

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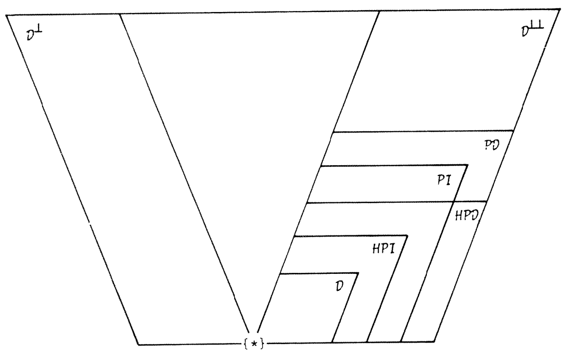

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